Extended Binet's formula for the class of generalized Fibonacci sequences

DIWAN Daksha M.
Department of Mathematics,
Government Engineering College,
Gandhinagar.
dm_diwan@yahoo.com

SHAH Devbhadra V.
Department of Mathematics,
Sir P.T.Sarvajanik College of Science,
Surat.
drdvshah@yahoo.com

Abstract

Fibonacci sequence $\{F_n\}$ *is defined by the recurrence relation* $F_n = F_{n-1} + F_{n-2}$ *for all* $n \geq 2$ *with initial condition* $F_0 = 0, F_1 = 1$. *This sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. Two of the generalizations of the Fibonacci sequence are the class of sequences* $\left\{H_n^{L(a,\,b)}\right\}$ *and* $\left\{H_n^{R(a,\,b)}\right\}$ *generated by the recurrence relation*

$$H_n^{L(a,\,b)} = \begin{cases} aH_{n-1}^{L(a,\,b)} + H_{n-2}^{L(a,\,b)} \;; when\ n\ is\ odd \\ H_{n-1}^{L(a,\,b)} + bH_{n-2}^{L(a,\,b)} \;; when\ n\ is\ even \end{cases} (n \geq 2)$$

and

$$H_n^{R(a,\,b)} = \begin{cases} H_{n-1}^{R(a,\,b)} + aH_{n-2}^{R(a,\,b)} \;; when\ n\ is\ odd \\ bH_{n-1}^{R(a,\,b)} + H_{n-2}^{R(a,\,b)} \;; when\ n\ is\ even \end{cases} (n \geq 2)$$

with initial condition $H_0^{L(a,\,b)} = H_0^{R(a,\,b)} = 0,\ H_1^{L(a,\,b)} = H_1^{R(a,\,b)} = 1$ *and* a, b *are positive integers. In this paper we obtain extended Binet's formula for the sequences* $\left\{H_n^{L(a,\,b)}\right\}$ *and* $\left\{H_n^{R(a,\,b)}\right\}$.

Key Words: Fibonacci sequence, generalized Fibonacci sequence, Binet formula.

1. Introduction:

The *Fibonacci sequence* $\{F_n\}$,named after Leonardo Pisano Fibonacci (1170−1250) is a sequence of numbers, starting with the integer pair 0 and 1, where the value of each element is calculated as the sum of the two preceding it. That is, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The first few terms of the Fibonacci sequence are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, … . We refer the readers to the book [1] for some basic background information and more details on the topic.

There are fundamentally two ways in which the Fibonacci sequence may be generalized; namely, either by maintaining the recurrence relation but altering the first two terms of the sequence from 0, 1 to arbitrary integers *a*, *b* or by preserving the first two terms of the sequence but altering the recurrence relation. The two techniques can be combined, but a change in the recurrence relation seems to lead to greater complexity in the properties of the resulting sequence. Many papers concerning a variety of generalizations of Fibonacci sequence have appeared in recent years. [2, 3]. Edson, Yayenie [4] generalized this sequence in to new class of generalized sequence which depends on two real parameters used in a recurrence relation. We define further generalizations of this sequence and call it the generalized Fibonacci sequences.

Definition: For any two positive numbers *a* and *b*, the generalized Fibonacci sequence $\left\{H_n^{L(a,b)}\right\}_{n=0}^{\infty}$ and $\left\{H_n^{R(a,b)}\right\}_{n=0}^{\infty}$ are the class of sequences defined recursively by

$$H_n^{L(a,b)} = \begin{cases} aH_{n-1}^{L(a,b)} + H_{n-2}^{L(a,b)} & ;\text{when } n \text{ is odd} \\ H_{n-1}^{L(a,b)} + bH_{n-2}^{L(a,b)} & ;\text{when } n \text{ is even} \end{cases} (n \geq 2)$$

and

$$H_n^{R(a,b)} = \begin{cases} H_{n-1}^{R(a,b)} + aH_{n-2}^{R(a,b)} & ;\text{when } n \text{ is odd} \\ bH_{n-1}^{R(a,b)} + H_{n-2}^{R(a,b)} & ;\text{when } n \text{ is even} \end{cases} (n \geq 2)$$

with initial condition $H_0^{L(a,b)} = H_0^{R(a,b)} = 0,\ H_1^{L(a,b)} = H_1^{R(a,b)} = 1.$

The Fibonacci sequence is a special case of these sequences with $a = b = 1$. In this paper we derive the extended Binet's formula for both $\left\{H_n^{L(a,b)}\right\}_{n=0}^{\infty}$ and $\left\{H_n^{R(a,b)}\right\}_{n=0}^{\infty}$.

2. Extended Binet's formula for $\left\{H_n^{L(a,b)}\right\}$:

We first note down two results which can be proved easily.

Lemma 2.1: For any positive integer *n*, the following holds:

$$1)\ H_{2n+4}^{L(a,b)} - (a+b+1)H_{2n+2}^{L(a,b)} + bH_{2n}^{L(a,b)} = 0\ .$$

$$2)\ H_{2n+5}^{L(a,b)} - (a+b+1)H_{2n+3}^{L(a,b)} + bH_{2n+1}^{L(a,b)} = 0.$$

We now obtain the value of two series related with $H_n^{L(a,b)}$.

Lemma 2.2: $\sum_{i=1}^{\infty} H_{2i}^{L(a,b)} x^{2i} = \frac{x^2}{bx^4-(a+b+1)x^2+1}$.

Proof: Let $p(x) = \sum_{i=1}^{\infty} H_{2i}^{L(a,b)} x^{2i} = H_2^{L(a,b)}x^2 + H_4^{L(a,b)}x^4 + H_6^{L(a,b)}x^6 + \cdots$.

Using Lemma 2.1 (1), we get $(bx^4 - (a+b+1)x^2 + 1)p(x) = x^2$. This gives

$$p(x) = \sum_{i=1}^{\infty} \mathrm{H}_{2i}^{\mathrm{L(a,\,b)}} x^{2i} = \frac{x^2}{bx^4-(a+b+1)x^2+1}.$$

Lemma 2.3: $\sum_{i=1}^{\infty} \mathrm{H}_{2i-1}^{\mathrm{L(a,\,b)}} x^{2i-1} = \frac{x-bx^3}{bx^4-(a+b+1)x^2+1}$.

Proof: Let $q(x) = \sum_{i=1}^{\infty} \mathrm{H}_{2i-1}^{\mathrm{L(a,\,b)}} x^{2i-1} = \mathrm{H}_1^{\mathrm{L(a,\,b)}} x + \mathrm{H}_3^{\mathrm{L(a,\,b)}} x^3 + \mathrm{H}_5^{\mathrm{L(a,\,b)}} x^5 + \cdots$.

Again using Lemma 2.1 (2), we get $(bx^4-(a+b+1)x^2+1)q(x) = x - bx^3$.

Thus we have $q(x) = \sum_{i=1}^{\infty} \mathrm{H}_{2i-1}^{\mathrm{L(a,\,b)}} x^{2i-1} = \frac{x-bx^3}{bx^4-(a+b+1)x^2+1}$.

We next derive the generating function for $\left\{\mathrm{H}_n^{\mathrm{L(a,\,b)}}\right\}$.

Lemma 2.4: The generating function for $\left\{\mathrm{H}_n^{\mathrm{L(a,\,b)}}\right\}$ is given by $h(x) = \frac{x+x^2-bx^3}{bx^4-(a+b+1)x^2+1}$.

Proof: Define

$$h(x) = \sum_{i=0}^{\infty} \mathrm{H}_i^{\mathrm{L(a,\,b)}} x^i = \mathrm{H}_0^{\mathrm{L(a,\,b)}} + \mathrm{H}_1^{\mathrm{L(a,\,b)}} x + \mathrm{H}_2^{\mathrm{L(a,\,b)}} x^2 + \mathrm{H}_3^{\mathrm{L(a,\,b)}} x^3 + \cdots. \quad (1)$$

After the rearrangement of terms, we get

$$(1-x-bx^2)h(x) = x + x\left(\sum_{i=1}^{\infty} \mathrm{H}_{2i}^{\mathrm{L(a,\,b)}} x^{2i}\right) + x^2\left(\sum_{i=1}^{\infty} \mathrm{H}_{2i-1}^{\mathrm{L(a,\,b)}} x^{2i-1}\right).$$

Using Lemma 2.2 and Lemma 2.3 and on simplification we get the generating function for $\left\{\mathrm{H}_n^{\mathrm{L(a,\,b)}}\right\}$ as $h(x) = \frac{x+x^2-bx^3}{bx^4-(a+b+1)x^2+1}$.

We are now all set to derive the extended Binet's formula for $\mathrm{H}_n^{\mathrm{L(a,\,b)}}$.

Theorem 2.5: The terms of the generalized Fibonacci sequence $\left\{\mathrm{H}_n^{\mathrm{L(a,\,b)}}\right\}$ are given by

$$\mathrm{H}_n^{\mathrm{L(a,\,b)}} = \frac{\gamma^{\chi(n)}\alpha^{[n/2]} - \bar{\gamma}^{\chi(n)}\beta^{[n/2]}}{\alpha-\beta}.$$

where $\alpha = \left(\frac{(a+b+1)+\sqrt{(a+b+1)^2-4b}}{2}\right)$, $\beta = \left(\frac{(a+b+1)-\sqrt{(a+b+1)^2-4b}}{2}\right)$ with $\gamma = \alpha - b$,

$\bar{\gamma} = \beta - b$ and $\chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}$.

Proof: Let α and β be the roots of equation $x^2-(a+b+1)x+b=0$.

This gives $\alpha = \left(\frac{(a+b+1)+\sqrt{(a+b+1)^2-4b}}{2}\right)$, $\beta = \left(\frac{(a+b+1)-\sqrt{(a+b+1)^2-4b}}{2}\right)$. Also we have $\alpha+\beta = (a+b+1)$, $\alpha\beta = b$.

Now by Lemma 2.4, we have $h(x) = \frac{x+x^2-bx^3}{bx^4-(a+b+1)x^2+1} = \frac{bx+bx^2-b^2x^3}{b^2x^4-b(a+b+1)x^2+b}$.

If we write $h(x) = \frac{bx+bx^2-b^2x^3}{b^2x^4-b(a+b+1)x^2+b} = \frac{(Ax+B)}{(bx^2-\alpha)} + \frac{(cx+D)}{(bx^2-\beta)}$ then by using the method of partial fractions, we get

$$h(x) = \frac{1}{(\alpha-\beta)}\left[\frac{-b(\alpha-1)x+\alpha}{(bx^2-\alpha)} + \frac{b(\beta-1)x-\beta}{(bx^2-\beta)}\right]. \quad (2)$$

But it is known that the Maclaurin's expansion of $\frac{P-Qx}{x^2-R}$ is given by

$$\frac{P-Qx}{x^2-R}=\sum_{n=0}^{\infty}QR^{-n-1}x^{2n+1}-\sum_{n=0}^{\infty}PR^{-n-1}x^{2n}.$$

Then $\frac{\alpha-b(\alpha-1)x}{b\left(x^2-\alpha/_b\right)}=\frac{1}{b}\left[\sum_{n=0}^{\infty}\frac{b(\alpha-1)}{\left(\alpha/_b\right)^{n+1}}x^{2n+1}-\sum_{n=0}^{\infty}\frac{\alpha}{\left(\alpha/_b\right)^{n+1}}x^{2n}\right]$ and

$$\frac{-(\beta-b(\beta-1)x)}{b\left(x^2-\beta/_b\right)}=-\frac{1}{b}\left[\sum_{n=0}^{\infty}\frac{b(\beta-1)}{\left(\beta/_b\right)^{n+1}}x^{2n+1}-\sum_{n=0}^{\infty}\frac{\beta}{\left(\beta/_b\right)^{n+1}}x^{2n}\right].$$

Using these in (2) we get

$$h(x)=\frac{1}{(\alpha-\beta)}\left[b^{n+1}\left\{\sum_{n=0}^{\infty}\left(\frac{\alpha-1}{\alpha^{n+1}}-\frac{\beta-1}{\beta^{n+1}}\right)x^{2n+1}-\sum_{n=0}^{\infty}\left(\frac{\alpha/_b}{\alpha^{n+1}}-\frac{\beta/_b}{\beta^{n+1}}\right)x^{2n}\right\}\right].$$

Since $\alpha\beta=b$, we get

$$h(x)=\frac{1}{(\alpha-\beta)}\{\sum_{n=0}^{\infty}\left((1-\beta)\alpha^{n+1}-(1-\alpha)\beta^{n+1}\right)x^{2n+1}+\sum_{n=0}^{\infty}(\alpha^n-\beta^n)x^{2n}\}.$$

But $(1-\beta)\alpha^{n+1}=(\alpha-b)\alpha^n$ and $(1-\alpha)\beta^{n+1}=(\beta-b)\beta^n$. For brevity we write $\gamma=\alpha-b$ and $\bar{\gamma}=\beta-b$. Thus we have

$$h(x)=\frac{1}{(\alpha-\beta)}\{\sum_{n=0}^{\infty}(\gamma\alpha^n-\bar{\gamma}\beta^n)x^{2n+1}-\sum_{n=0}^{\infty}(\alpha^n-\beta^n)x^{2n}\}.$$

On defining $\chi(n)=1$, if n is odd and $\chi(n)=0$, if n is even we write $h(x)$ as

$$h(x)=\sum_{n=0}^{\infty}\frac{\gamma^{\chi(n)}\alpha^{[n/_2]}-\bar{\gamma}^{\chi(n)}\beta^{[n/_2]}}{\alpha-\beta}x^n. \quad (3)$$

Finally by (1) and (3) we have

$$\mathrm{H}_n^{L(a,\,b)}=\frac{\gamma^{\chi(n)}\alpha^{[n/_2]}-\bar{\gamma}^{\chi(n)}\beta^{[n/_2]}}{\alpha-\beta}.$$

3. Extended Binet's formula for $\left\{\mathrm{H}_n^{R(a,\,b)}\right\}$:

In this section we use the techniques similar to that used in the above section.

Lemma 3.1: For any positive integer *n*, the following holds:

1) $\mathrm{H}_{2n+4}^{R(a,b)}-(a+b+1)\mathrm{H}_{2n+2}^{R(a,b)}+a\mathrm{H}_{2n}^{R(a,b)}=0$.

2) $\mathrm{H}_{2n+5}^{R(a,b)}-(a+b+1)\mathrm{H}_{2n+3}^{R(a,b)}+b\mathrm{H}_{2n+1}^{R(a,b)}=0$.

Here too we obtain the value of two series related with $\mathrm{H}_n^{R(a,\,b)}$.

Lemma 3.2: $\sum_{i=0}^{\infty}\mathrm{H}_{2i}^{R(a,\,b)}x^{2i}=\frac{bx^2}{ax^4-(a+b+1)x^2+1}$.

Proof: Let $u(x)=\sum_{i=0}^{\infty}\mathrm{H}_{2i}^{R(a,\,b)}x^{2i}=\mathrm{H}_0^{R(a,\,b)}+\mathrm{H}_2^{R(a,b)}x^2+\mathrm{H}_4^{R(a,\,b)}x^4$
$+\mathrm{H}_6^{R(a,b)}x^6+\cdots$.

Using Lemma 3.1 (1), we get $(ax^4-(a+b+1)x^2+1)u(x)=bx^2$, which gives

$$u(x)=\sum_{i=0}^{\infty}\mathrm{H}_{2i}^{R(a,\,b)}x^{2i}=\frac{bx^2}{ax^4-(a+b+1)x^2+1}.$$

Lemma 3.3: $\sum_{i=0}^{\infty} \mathrm{H}_{2i-1}^{R(a,b)} x^{2i+1} = \frac{x-x^3}{ax^4-(a+b+1)x^2+1}$.

Proof: Let $v(x) = \sum_{i=0}^{\infty} \mathrm{H}_{2i+1}^{R(a,b)} x^{2i-1} = \mathrm{H}_1^{R(a,b)} x + \mathrm{H}_3^{R(a,b)} x^3 + \mathrm{H}_5^{R(a,b)} x^5 + \cdots$

By using Lemma 3.1 (2), we get $(ax^4 - (a+b+1)x^2 + 1)v(x) = x - x^3$.

Thus we get $v(x) = \sum_{i=0}^{\infty} \mathrm{H}_{2i+1}^{R(a,b)} x^{2i-1} = \frac{x-x^3}{ax^4-(a+b+1)x^2+1}$.

We now derive the generating function for $\left\{\mathrm{H}_n^{R(a,b)}\right\}$.

Lemma 3.4: The generating function for $\left\{\mathrm{H}_n^{R(a,b)}\right\}$ is given by $l(x) = \frac{x+bx^2-x^3}{ax^4-(a+b+1)x^2+1}$.

Proof: Define

$$l(x) = \sum_{i=0}^{\infty} \mathrm{H}_i^{R(a,b)} x^i = \mathrm{H}_0^{R(a,b)} + \mathrm{H}_1^{R(a,b)} x + \mathrm{H}_2^{R(a,b)} x^2 + \mathrm{H}_3^{R(a,b)} x^3 + \cdots. \quad (4)$$

After the rearrangement of terms, we get

$$(1 - x - ax^2)l(x) = x + x^2(1-a)\left(\sum_{i=0}^{\infty} \mathrm{H}_{2i}^{R(a,b)} x^{2i}\right) + x(b-1)\left(\sum_{i=0}^{\infty} \mathrm{H}_{2i+1}^{R(a,b)} x^{2i+1}\right)$$

Using Lemma 3.2, 3.3 and on simplification we get the generating function for $\left\{\mathrm{H}_n^{R(a,b)}\right\}$ as $l(x) = \frac{x+bx^2-x^3}{ax^4-(a+b+1)x^2+1}$.

We finally obtain the extended Binet's formula for $\mathrm{H}_n^{R(a,b)}$.

Theorem 3.5: The terms of the generalized Fibonacci sequence $\left\{\mathrm{H}_n^{R(a,b)}\right\}$ are given by

$$\mathrm{H}_n^{R(a,b)} = b^{1-\chi(n)}\left(\frac{\gamma^{\chi(n)}\alpha^{[n/2]} - \bar{\gamma}^{\chi(n)}\beta^{[n/2]}}{\alpha-\beta}\right).$$

where $\alpha = \left(\frac{(a+b+1)+\sqrt{(a+b+1)^2-4a}}{2}\right)$, $\beta = \left(\frac{(a+b+1)-\sqrt{(a+b+1)^2-4a}}{2}\right)$ with $\gamma = \alpha - 1$,

$\bar{\gamma} = \beta - 1$ and $\chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}$.

Proof: Let α and β be the roots of equation $x^2 - (a+b+1)x + a = 0$.

This gives $\alpha = \left(\frac{(a+b+1)+\sqrt{(a+b+1)^2-4a}}{2}\right)$, $\beta = \left(\frac{(a+b+1)-\sqrt{(a+b+1)^2-4a}}{2}\right)$. Also we have $\alpha + \beta = (a+b+1)$, $\alpha\beta = a$.

Using Lemma 3.4, we have $l(x) = \frac{x+bx^2-x^3}{ax^4-(a+b+1)ax^2+a} = \frac{ax+abx^2-ax^3}{a^2x^4-a(a+b+1)x^2+a}$.

If we write $l(x) = \frac{ax+abx^2-ax^3}{a^2x^4-a(a+b+1)x^2+a} = \frac{(Ax+B)}{(ax^2-\alpha)} + \frac{(cx+D)}{(ax^2-\beta)}$ then by using the method of partial fractions, we get

$$l(x) = \frac{1}{(\alpha-\beta)}\left[\frac{(\alpha-a)x+\alpha b}{(ax^2-\alpha)} + \frac{(\beta-a)x-b\beta}{(ax^2-\beta)}\right]. \quad (5)$$

Now by using Maclaurin's expansion we get

$$\frac{\alpha b-(\alpha-a)x}{a(x^2-\alpha/a)} = \frac{1}{a}\left[\sum_{n=0}^{\infty}\frac{\alpha-a}{(\alpha/a)^{n+1}}x^{2n+1} - \sum_{n=0}^{\infty}\frac{\alpha b}{(\alpha/a)^{n+1}}x^{2n}\right] \text{ and}$$

$$\frac{-b\beta-(a-\beta)x}{b\left(x^2-\beta/b\right)} = -\frac{1}{a}\left[\sum_{n=0}^{\infty}\frac{a-\beta}{\left(\beta/a\right)^{n+1}}x^{2n+1} - \sum_{n=0}^{\infty}\frac{-b\beta}{\left(\beta/a\right)^{n+1}}x^{2n}\right].$$

Using these in (5) we get

$$l(x) = \frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{n+1}}{a}\left\{\sum_{n=0}^{\infty}\left(\frac{\alpha-a}{\alpha^{n+1}}-\frac{a-\beta}{\beta^{n+1}}\right)x^{2n+1}-\sum_{n=0}^{\infty}\left(\frac{\alpha b}{\alpha^{n+1}}-\frac{b\beta}{\beta^{n+1}}\right)x^{2n}\right\}\right].$$

Using $\alpha\beta = a$, we get

$$l(x) = \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left((\alpha-1)\alpha^n-(\beta-1)\beta^n\right)x^{2n+1}+\sum_{n=0}^{\infty}b(\alpha^n-\beta^n)x^{2n}\right\}.$$

Writing $\gamma = \alpha - 1$ and $\bar{\gamma} = \beta - 1$, we have

$$l(x) = \frac{1}{(\alpha-\beta)}\{\sum_{n=0}^{\infty}(\gamma\alpha^n-\bar{\gamma}\beta^n)x^{2n+1}-\sum_{n=0}^{\infty}b(\alpha^n-\beta^n)x^{2n}\}.$$

$$= \sum_{n=0}^{\infty} b^{1-\chi(n)}\left(\frac{\gamma^{\chi(n)}\alpha^{\lceil n/2\rceil}-\bar{\gamma}^{\chi(n)}\beta^{\lceil n/2\rceil}}{\alpha-\beta}\right)x^n.$$

This gives

$$\mathrm{H}_{\mathrm{n}}^{\mathrm{R(a,\,b)}} = b^{1-\chi(n)}\left(\frac{\gamma^{\chi(n)}\alpha^{\lceil n/2\rceil}-\bar{\gamma}^{\chi(n)}\beta^{\lceil n/2\rceil}}{\alpha-\beta}\right).$$

References:

[1] T. Koshy, Fibonacci and Lucas Numbers with applications, Pure and Applied Mathematics, Wiley-Inter science, New York, NY, USA, 2001.

[2] De Villiers, M., A Fibonacci generalization and its dual, International Journal of Mathematics Education, Science and Technology, 31, 2000, 464-473.

[3] Falcon S., A simple proof of an interesting Fibonacci generalization, International Journal of Mathematics Education, Science and Technology, 35, No. 2,2004, 259-261.

[4] Marcia Edson, Omer Yayenie, A new generalization of Fibonacci sequence and extended Binet's formula, Integers, Volume 9, Issue 6, 2009, 639–654.