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Closed form continued fraction expansions for the powers of Lucas golden ratio

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Abstract
We consider the Lucas sequence $\{L_n\}$ and define the Lucas golden proportion as $\phi = \lim_{n\to\infty} \frac{L_n}{L_{n-1}}$. By using properties of the recurrence relation and continued fractions, we find simple closed form continued fraction expansions for ϕ^k, for any positive integer k.

Key Words: Fibonacci sequence, Lucas sequence, Continued fraction, Golden proportion.

1. Introduction:

Continued fractions provide deep insight into mathematical problems; particularly into the nature of numbers. Continued fractions have found applications in various areas of Physics such as Fabry-Perot interferometry, quasi-amorphous states of matter and chaos. It encodes much useful information about the algebraic structure of a number and frequently arises in approximation theory and dynamical systems. Van der Poorten [1] wrote that the elementary nature and simplicity of the theory of continued fractions is mostly buried in the literature. Our work is an outgrowth of [1, 2, 3]. We refer the readers to these papers for some basic background information on continued fractions, and to the book [4] for more details.

It is known that every real number α has a continued fraction expansion

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} ,$$

where each a_i is an integer (and a positive integer unless $i = 0$). For brevity we write $\alpha = [a_0, a_1, a_2, \ldots]$.

Clearly, α is rational if and only if its continued fraction is finite, and a beautiful theorem of Lagrange asserts that α is a quadratic irrational if and only if the continued fraction expansion is periodic.

The continued fraction expansion consisting of the number 1 repeated indefinitely represents the '*golden mean*'. This satisfies the quadratic equation $x^2 = x + 1$. The convergents of this continued fraction are obtained as the ratio of the successive terms of the Fibonacci sequence. Recall that the Fibonacci sequence $\{F_n\}$, named after Leonardo Pisano Fibonacci (1170−1250), is defined as $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2};\ n \geq 2$, which gives the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … .

The Lucas numbers $\{L_n\}$, named after François Lucas (1842−1891), are defined by

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}; n \geq 2.$$

First few terms of the sequence are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ...

The Binet's formula for $\{L_n\}$ is given by $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$. It can be easily proved that $\lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \frac{1+\sqrt{5}}{2} = [1,1,1,1,\ldots](= \phi)$. In this paper we derive the closed form expressions for the continued fraction expansions for ϕ^k, for any fixed positive integer *k*. In fact, we prove the following result:

Theorem 1: $\frac{L_n}{L_{n-k}} = \begin{cases} \left[L_k, \frac{L_{n-k}}{L_{n-2k}}\right] & ;\ \text{if } k \text{ is odd} \\ \left[L_k - 1, 1, \frac{L_{n-k}}{L_{n-2k}} - 1\right]; & \text{if } k \text{ is even,} \end{cases}$

which yields

$$\phi^k = \begin{cases} [\overline{L_k, L_k}] & ;\text{if } k \text{ is odd} \\ [L_k - 1, \overline{1, L_k - 2}]; & \text{if } k \text{ is even.} \end{cases}$$

2. Continued Fractions of powers of the Golden Mean ϕ:

The following lemma collects some well-known properties of the Fibonacci numbers and Lucas numbers which will be useful below.

Lemma 2: The Lucas numbers satisfy the following properties:

1) $L_n = F_{n-1} + F_{n+1}$.
2) $F_{k-2}F_k - (F_{k-1})^2 = (-1)^{k-1}$.

3) $L_n = L_{n-k}F_{k+1} + L_{n-k-1}F_k$.

4) $\phi = \lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \frac{1+\sqrt{5}}{2}$.

The proofs of all these results are standard and can be found in [5], especially in Chapter 5. Still we prove the fourth result for the sake of continuity of the topic.

We have $\frac{L_n}{L_{n-1}} = \frac{L_{n-1}+L_{n-2}}{L_{n-1}} = 1 + \frac{L_{n-2}}{L_{n-1}} = 1 + \frac{1}{\frac{L_{n-1}}{L_{n-2}}}$.

As $n \to \infty$, this yield the equation $x = 1 + \frac{1}{x} \Rightarrow x^2 = x + 1$. Solving this quadratic equation we get $x = \frac{1\pm\sqrt{5}}{2}$. Since the limit is positive, it follows that

$$\phi = \lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \frac{1+\sqrt{5}}{2}, \text{ as required.}$$

We note that for the golden mean ϕ, we have $\phi = \frac{1+\sqrt{5}}{2} = [1,1,1,1,1, \ldots]$. This gives

$$\lim_{n\to\infty} \frac{L_n}{L_{n-k}} = \lim_{n\to\infty} \frac{L_n}{L_{n-1}} \frac{L_{n-1}}{L_{n-2}} \ldots \frac{L_{n-(k-1)}}{L_{n-k}} = \phi^k.$$

For $k = 2$, it is trivial to find the continued fraction of ϕ^2, since $L_n = L_{n-1} + L_{n-2}$ implies

$$\frac{L_n}{L_{n-2}} = 1 + \frac{L_{n-1}}{L_{n-2}}.$$

Taking the limit as $n \to \infty$ we get

$$\phi^2 = \lim_{n\to\infty} \frac{L_n}{L_{n-2}} = 1 + \phi = [2,1,1,1, \ldots].$$

With $\mathrm{k} = 3$, however, the result is not as obvious. From Lemma 2 we have

$$L_n = L_{n-2}F_3 + L_{n-3}F_2 = 2\,L_{n-2} + L_{n-3}$$

$\therefore \frac{L_n}{L_{n-3}} = 1 + 2\frac{L_{n-2}}{L_{n-3}}$.

Unfortunately, for any given expansion of α , in general there is no simple expression for the continued fraction of 2α.

However, $\frac{L_n}{L_{n-3}} = \frac{L_{n-1}+L_{n-2}}{L_{n-3}} = \frac{L_{n-2}+L_{n-3}+L_{n-2}}{L_{n-3}} = \frac{2L_{n-2}+L_{n-3}}{L_{n-3}} = 1 + 2\frac{L_{n-2}}{L_{n-3}}$

$$= 1 + \frac{2L_{n-3}+2L_{n-4}}{L_{n-3}} = 3 + 2\frac{L_{n-4}}{L_{n-3}} = 3 + \frac{L_{n-4}+L_{n-3}-L_{n-5}}{L_{n-3}}$$

$$= 4 + \frac{L_{n-4}-L_{n-5}}{L_{n-3}}$$

Thus $\frac{L_n}{L_{n-3}} = 4 + \frac{L_{n-6}}{L_{n-3}}$.

Numerical computations of the continued fraction expansions of $\frac{L_n}{L_{n-3}}$ were found to have the form $[4,4, \ldots, t]$, where $t = \begin{cases} 4, \text{if } \mathrm{n} \equiv 0 (\mathrm{mod}\ 3) \\ 3, \text{if } \mathrm{n} \equiv 1 (\mathrm{mod}\ 3) \\ 5, \text{if } \mathrm{n} \equiv 2 (\mathrm{mod}\ 3) \end{cases}$.

Now knowing the first few ratios, we get the continued fraction expansion for Φ^3 simply by taking limits.

Theorem 1 states that this algebra can be generalized to any k, as we now show.

Proof of Theorem 1: We have,

$$\frac{L_n}{L_{n-k}} = \frac{F_{k+1}L_{n-k} + F_k\,L_{n-k-1}}{L_{n-k}}$$
$$= F_{k+1} + F_k\,\frac{L_{n-k-1}}{L_{n-k}}$$
$$= F_{k+1} + \frac{F_{k-1}L_{n-k-1} + F_{k-2}\,L_{n-k-1}}{L_{n-k}}$$
$$= F_{k+1} + \frac{F_{k-1}\,L_{n-k-1} + F_{k-1}\,L_{n-k-2} + F_{k-2}\,L_{n-k-1} - F_{k-1}\,L_{n-k-2}}{L_{n-k}}$$
$$= F_{k+1} + F_{k-1}\frac{L_{n-k}}{L_{n-k}} + \frac{F_{k-2}L_{n-k-1} - F_{k-1}\,L_{n-k-2}}{L_{n-k}}$$
$$= F_{k+1} + F_{k-1} + \frac{F_{k-2}\,L_{n-k-1} - F_{k-1}\,L_{n-k-2}}{L_{n-k}}$$
$$= L_k + \frac{F_{k-2}L_{n-k-1} - F_{k-1}\,L_{n-k-2}}{L_{n-k}}.$$

Now by using Lemma 2, we get $L_{n-k-1} = F_k\,L_{n-k-k} + F_{k-1}\,L_{n-k-k-1}$ and $L_{n-k-2} = F_{k-1}\,L_{n-k-k-1} + F_{k-2}\,L_{n-k-k-2}$. This gives

$$\frac{L_n}{L_{n-k}}$$
$$= L_k + \frac{(F_{k-2}(F_k\,L_{n-2k} + F_{k-1}\,L_{n-2k-1}) - F_{k-1}(F_{k-1}\,L_{n-2k} + F_{k-2}\,L_{n-2k-1}))}{L_{n-k}}$$
$$= L_k + \frac{F_{k-2}(F_k\,L_{n-2k}) - F_{k-1}(F_{k-1}\,L_{n-2k})}{L_{n-k}}$$
$$= L_k + \frac{L_{n-2k}[F_{k-2}F_k - (F_{k-1})^2]}{L_{n-k}}.$$

Once again using Lemma 2, we get

$$\frac{L_n}{L_{n-k}} = L_k + \frac{L_{n-2k}\,(-1)^{k-1}}{L_{n-k}}.$$

This gives

$$\frac{L_n}{L_{n-k}} = L_k + \frac{(-1)^{k-1}}{L_{n-k}/L_{n-2k}}.$$

Now if k is odd then we get

$$\frac{L_n}{L_{n-k}} = L_k + \frac{1}{L_{n-k}/L_{n-2k}}.$$

Also if k is even then we get

$$\frac{L_n}{L_{n-k}} = L_k + \frac{-1}{L_{n-k}/L_{n-2k}}.$$

In this case we manipulate further. We write it as

$$\frac{L_n}{L_{n-k}} = (L_k - 1) + 1 - \frac{1}{L_{n-k}/L_{n-2k}}.$$

Now,

$$1 - \frac{1}{L_{n-k}/L_{n-2k}} = \frac{L_{n-k} - L_{n-2k}}{L_{n-k}}$$
$$= \frac{1}{L_{n-k}/(L_{n-k} - L_{n-2k})}$$

$$= \frac{1}{((L_{n-k} - L_{n-2k}) + L_{n-2k})/(L_{n-k} - L_{n-2k})}$$
$$= \frac{1}{1+(L_{n-2k}/(L_{n-k} - L_{n-2k}))}$$
$$= \frac{1}{1+(1/((L_{n-k} - L_{n-2k})/L_{n-2k}))}$$
$$= \frac{1}{1+(1/((L_{n-k}/L_{n-2k})-1))}$$

Taking the limit as $n \to \infty$, we get

$$\phi^k = L_k - 1 + \frac{1}{1+(1/(\phi^k-1))}.$$

Finally the continued fraction of ϕ^k follows easily.

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