2203080301010002
Examination February 2024
MASTER OF SCIENCE (MATHEMATICS) (FIRST SEMESTER)
ADVANCED REAL ANALYSIS

[Time: Three Hours] **[Max. Marks: 70]**

Instructions:
1. Fill up strictly the following details on your answer book
   a. Name of the Examination: **MASTER OF SCIENCE (MATHEMATICS) (FIRST SEMESTER)**
   b. Name of the Subject: **ADVANCED REAL ANALYSIS**
   c. Subject Code No: **2203080301010002**
2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. All questions are compulsory.
5. Follow usual notations.

Seat No:

Student's Signature

Q.1 Attempt any two: (7 marks each) **14**

1. Given any collection C of subsets of X, prove that there exists a σ-algebra A containing C such that if B is any σ -algebra containing C, then $A \subset B$.

2. Define Borel sets. Prove that F_σ and G_δ sets are Borel sets. Also justify that every open set is an F_σ set.

3. Define open set and closed set. Prove that the complement of an open set is closed and the complement of a closed set is open.

Q.2 Attempt any two: (7 marks each) **14**

1. Prove that the outer measure of an interval is its length.

2. Define measurable set. If m* (E) =0, then prove that E is measurable. Further if E_1 and E_2 are measurable sets then prove that $E_1 \cup E_2$ is measurable.

3. Let $\langle E_i \rangle$ be a sequence of measurable sets. If the sets E_i are pairwise disjoint, then prove that $m(\bigcup E_i) = \sum m\, E_i$.

Q.3 Attempt any two: (7 marks each) **14**

1. Prove that the set E is measurable if and only if given $\epsilon > 0$, there is a closed set $F \subset E$ aith $m * (E \sim F) < \epsilon$.

2. Let c be any real constant and f, g be two measurable real valued functions defined on the same domain, then prove that the functions $f + c, cf, f + g, g - f$ are also measurable.

3. Let $\langle f_n \rangle$ be a sequence of measurable functions defined on same domain. Prove that the functions $\sup\{f_1, f_2, \ldots, f_n\}, \sup_n f_n and\ \overline{lim}\ f_n$ are all measurable.

Q.4 Attempt any two: (7 marks each) **14**

1. Define Riemann Integral and Riemann integrable functions. Express Riemann lower integral and Riemann upper integral in terms of step function.

2. Define simple function. Let φ and ψ be simple functions which vanish outside a set of finite measure. Prove that $\int (a\varphi + b\psi) = a \int \varphi + b \int \psi$.

3. Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure and suppose that $|f_n(x)| \leq M\ for\ all\ n\ and\ all\ x$. If $f(x) = \lim f_n(x)$ for each x in E, then prove that $\int_E f = \lim \int_e f_n$.

Q.5 Attempt any two: (7 marks each) **14**

1. State Fatou's lemma. If $\langle f_n \rangle$is an increasing sequence of non-negative measurable functions, and $f = \lim f_n$ a.e., then prove that $\int f = \lim \int f_n$.

2. Let f be a non-negative function which is integrable over a set E. Prove that given $\epsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta we$ have $\int_A f < \epsilon$.

3. Define integrable function. Let f and g be integrable over E. Prove that:
   a) The function cf is integrable over E, and $\int_E cf = c\int_E f$.
   b) The function $f + g$ is integrable over E, and $\int_E f + g = \int_E f + \int_E g$.

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