2303080302020002
MASTER OF SCIENCE (MATHEMATICS)
(SECOND SEMESTER)
COMPLEX ANALYSIS

[**Time: Three Hours]** **[Max. Marks: 70]**

Instructions:

1. **Fill up strictly the following details on your answer book**
   a. Name of the Examination: **MASTER OF SCIENCE (MATHEMATICS) (SECOND SEMESTER)**
   b. Name of the Subject: **COMPLEX ANALYSIS**
   c. Subject Code No: **2303080302020002**
2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. All questions are compulsory.

Seat No:

Student's Signature

Q.1 Attempt Any Two: **14**

1. Let the function $f(z) = u(x,y) + iv(x,y)\ and\ f'(z)\ exists\ at\ z_0 = x_0 + iy_0$. Then show that, the first order partial derivatives of u and v at (x_0, y_0) satisfy $u_x = v_y$ and $u_y = -v_x$. Moreover show that, $f'(z_0) = u_x + iv_x\ or\ f'(z_0) = v_y - iu_y$.

2. Define harmonic function and Harmonic conjugate. Also show that, if a function $f(z) = u + iv$ is an analytic in domain D then its component functions u and v are harmonic in D.

3. Show that the function $u(x,y) = cos\ x\ sinh\ y$ is harmonic and find its harmonic conjugate and construct corresponding analytic function.

Q.2 Attempt Any Two: **14**

1. State ML-inequality. Also show that $\left|\int_C \frac{z^{-i}}{z^2-1} dz\right| \leq \frac{4\pi e^{2\pi}}{3}$ where C denote the circle, $|z| = 2$, described in counter clockwise direction and Consider the branch $\log z = ln\,|z| + i\theta\ (|z| > 0, 0 < \theta < 2\pi)$.

2. Let $f(z)$ be analytic in a domain D and f' is continuous in D. Then prove that, $\int_y f(z)dz = 0$ for every simple closed contour γ in D.

3. By using Cauchy integral formula, evaluate the integral, $\int_c \frac{\sinh z}{z^2(z-2)}\ dz$, where C ={z: |z|=1}, taken in the positive sense.

Q.3 Attempt Any Two: **14**

1. If $\sum_{n=0}^{\infty} a_n$ converges, then prove that, the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with R=1 tends to $f(1) as\ z \to 1$, provided $|1-z|/(1-|z|)$ remains bounded.

2. Let f(z) be analytic in $|z| < 1$, with a zero of order n at the origin. Suppose that $|f(z)| \leq 1$for all z in |z| <1. Then prove that,
   $|f(z)| \leq |z|^n, |z| < 1$ ---------- (1)
   and $|f^n(0)| \leq n!$ ---------- (2)
   Further show that, equality holds in (1) for some $z \neq 0$ or in (2) if and only if $f(z)$ is of the form $f(z) = cz^n, |c| = 1$

3. Derive the Laurent series of the function $f(z) = \frac{z}{(z+a)(z+b)}, |a| < |b|$ in powers of z.

Q.4 Attempt Any Two: **14**

1. Let $f(z)$ be an analytic function at z_0 then show that, the function f(z) has a pole of order m at z_0 if and only if $f(z)$ can be written in the form.
   $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, where $\phi(z_0) \neq 0$ and $\phi(z)$ is analytic at z_0.

2. If γ is a positively oriented simple closed contour within and on which function is analytic, except at a finite number of singularities $z_k (k = 1,2, \dots \dots, n)$ interior to γ, then prove that $\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k}\ f(z)$.

3. Evaluate the integral, $\int_y \frac{z}{(z^2+1)(z-3)^2} dz$, by using Cauchy-residue theorem, where $\gamma = \{z: |z| = 2\}$ positively oriented.

Q.5 Attempt Any Two: **14**

1. Suppose that a function f is analytic in the annular domain $R_1 < |z-a| < R_2$. Let C denote any positively oriented circle around a and lying in this domain. Then prove that, $f(z)$ can be expanded about any point in the annular domain. $R_1 < |z-a| < R_2, f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$,

2. Where $a_n = \frac{1}{2\pi i} \int_c \frac{f(s)}{(s-a)^{n+1}}\ ds$, $n = 0, 1, 2, \dots$ and $b_n = \frac{1}{2\pi i} \int_c \frac{f(s)}{(s-a)^{-n+1}}\ ds$, $n = 1,2,3, \dots$

3. Show that, f(z) = $\begin{cases} \frac{(\bar{z})^2}{z}, z \neq 0 \\ 0, \quad z = 0 \end{cases}$ Satisfy C-R equations at (0,0). Is the function differentiable at z=0 ?

4. Decide the nature of all singularities of the function $\frac{\pi \cot \pi z}{z^2}$. Also, Calculate the residue at these singularities.

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