2303080304040006
EXAMINATION APRIL 2025
MASTER OF SCIENCE (MATHEMATICS) (FOURTH SEMESTER)
ADVANCED SPECIAL FUNCTIONS - II

[Time: As Per Schedule] **[Max. Marks: 70]**

Instructions:
1. Fill up strictly the following details on your answer book
   a. Name of the Examination : **MASTER OF SCIENCE (MATHEMATICS) (FOURTH SEMESTER)**
   b. Name of the Subject : **ADVANCED SPECIAL FUNCTIONS - II**
   c. Subject Code No : **2303080304040006**
2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. All questions are compulsory.
5. Follow Usual notations and conventions.

Seat No:

Student's Signature

Q.1 Answer Any <u>Two</u> of the following. **14**

I. Define Laguerre polynomials. Derive the Orthogonal Property of Laguerre Polynomials.

II. Prove that

$$L_n^{(\propto)}(x) = \frac{(1+\propto)_n}{(c)_n}\sum_{k=0}^{\infty}\frac{(1+\propto-c)_n L_n^{(2c-\propto-2)}(x) L_n^{(\propto)}(x)}{(1+\propto)_n(1+\beta)_n} .$$

III. Prove that

$$L_n^{(\propto)}(xy) = \sum_{k=0}^{\infty}\frac{(1+\propto)_n(1-y)^{n-k}y^k \; L_k^{(\propto)}(-x)}{(n-k)!(1+\propto)_k}.$$

Q.2 Answer any <u>Two</u> of the following. **14**

I. With usual notation prove that.

$$nL_n^{(\propto)}(x) = (2n-1+\alpha-x)L_{n-1}^{(\propto)}(x) - (n-1+\alpha)L_{n-2}^{(\propto)}(x)$$

II. State and prove the Rodrigues formula for $L_n^{(\propto)}(x)$.

III. Obtain the Recurrence relations

$$\text{i)} x\, D\, L_n^{(\propto)}(x) = n\, L_n^{(\propto)}(x) - (\alpha+n)L_{n-1}^{(\propto)}(x)$$

$$\text{ii)} D\, L_n^{(\propto)}(x) = -\sum_{k=0}^{n-1} L_k^{(\propto)}(x)$$

Q.3 Answer any Two of the following. **14**

I. With usual Notation prove that

$$p_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_{2n}}{(1+\alpha+\beta)_n\, n!}\left(\frac{x-1}{\alpha}\right)^n \; {}_2F_1\left[\begin{matrix}-n,\alpha-n;\\ -\alpha-\beta-2n;\end{matrix}\frac{2}{1-x}\right]$$

II. With usual notation prove that

$$(x-1)\left[(\alpha+\beta+n)DP_n^{(\alpha,\beta)}(x) + (\alpha+n)DP_{n-1}^{(\alpha,\beta)}(x)\right]$$
$$= (\alpha+\beta+n)\left[nP_n^{(\alpha,\beta)}(x) - (\alpha+n)P_{n-1}^{(\alpha,\beta)}(x)\right]$$

III. With usual Notation Show that

$$\sum_{k=0}^{\infty}\frac{(\gamma)_n(1+\alpha+\beta-\gamma)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n(1+\beta)_n}t^n$$
$$= {}_2F_1\left[\begin{matrix}\gamma, & 1+\alpha+\beta-\gamma;\\ & 1+\alpha;\end{matrix}\frac{1-t-\rho}{2}\right]\; {}_2F_1\left[\begin{matrix}\gamma, & 1+\alpha+\beta-\gamma;\\ & 1+\beta;\end{matrix}\frac{1+t-\rho}{2}\right]$$

Q.4 Answer any Two of the following. **14**

I. With Usual Notation Prove that

$$(x-1)DP_n^{(\alpha,\beta)}(x) - nP_n^{(\alpha,\beta)}(x) =$$
$$\frac{(1+\alpha)_n}{(1+\alpha+\beta)_n}\sum_{k=0}^{n-1}(-1)^{n-k}\frac{(1+\alpha+\beta)_k}{(1+\alpha)_k}\,(1+\alpha+\beta+2k)P_k^{(\alpha,\beta)}(x)$$

II. With usual Notation prove that

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n\,(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n\, n!}D^n\left[(1-x)^{n+\alpha}(1+x)^{n-\beta}\right]$$

III. $\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}P_k^{(\alpha,\beta)}(x)t^n$

$$= \frac{1}{(1-t)^{1+\alpha+\beta}}\;{}_2F_1\left[\begin{matrix}\frac{1+\alpha+\beta}{2}, & \frac{2+\alpha+\beta}{2};\\ & 1+\alpha;\end{matrix}\frac{2t(x-1)}{(1-t)^2}\right]$$

Q.5 **Answer any Two of the following.** **14**

I. Define Weierstrass Function P(z).Prove that P(z) is an elliptic function with poles of order two.

II. Prove that

$L_n^{(\propto+\beta+1)}(x+y) = \sum_{k=0}^{n} L_k^{(\propto)}(x) L_{n-k}^{(\beta)}(y)$.

III. Prove that

$\sum_{k=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x)t^n}{(1+\alpha)_n(1+\beta)_n} = {}_0F_1\left[\begin{matrix}-;\\ 1+\alpha;\end{matrix}\frac{t(x-1)}{\alpha}\right] {}_0F_1\left[\begin{matrix}-;\\ 1+\beta;\end{matrix}\frac{t(x+1)}{2}\right]$.

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